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NUMBERS WHOSE POSITIVE DIVISORS HAVE SMALL INTEGRAL HARMONIC MEAN

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To Peter Hagis, Jr., on the occasion of his 70th birthday

ABSTRACT. A natural number n is said to be harmonic when the harmonic mean $H(n)$ of its positive divisors is an integer. These were first introduced almost fifty years ago. In this paper, all harmonic numbers less than 2×10^9 are listed, along with some other useful tables, and all harmonic numbers n with $H(n) \leq 13$ are determined.

1.

Let $\tau(n)$ and $\sigma(n)$ denote the number of positive divisors of a positive integer n, and their sum, respectively. The harmonic mean of these divisors is easily seen to be

$$H(n) = \frac{n\tau(n)}{\sigma(n)}.$$

Then n is said to be harmonic if $H(n)$ is an integer. Harmonic numbers were first studied by Ore [7], and they remain of interest because of their connection with perfect numbers. Recall that n is perfect if $\sigma(n) = 2n$; it is easy to show that every perfect number is harmonic.

A list of the harmonic numbers less than $2 \cdot 10^9$ is given in Table 3, at the end of this paper. This extends the lists of Ore [7] and Garcia [3], which gave all harmonic numbers up to 10^5 and 10^7, respectively. We see that no nontrivial example of an odd harmonic number is known; if it could be proved that in fact there are none, then this would imply the nonexistence of odd perfect numbers.

In [4] Guy wrote: "Which values does the harmonic mean take? Presumably not 4, 12, 16, 18, 20, 22, ... ; does it take the value 23?" We have settled the first of these questions for the first two values in Theorem 3, below. This paper gives only a brief sketch of the proofs of the various results. Full details are given in [2].

Theorem 1. *The only harmonic numbers of the form $2^a m$, where m is odd and squarefree and $1 \leq a \leq 11$, are those listed in Tables* 1a *and* 1b. (*There are* 52 *such numbers including* 45 *when* $a = 8$.)

Theorem 2. *Let $\omega(n)$ denote the number of distinct prime factors of n. For all n,*

$$H(n) > \frac{2^{\omega(n)+1}}{\omega(n) + 1}, \tag{1}$$

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TABLE 1a

a	*All* $2^a m \in \mathcal{H}$ *with* m *odd and squarefree*
1	$2 \cdot 3 = 6$
2	$2^2 7 = 28$, $2^2 5 \cdot 7 = 140$
3	none
4	$2^4 31 = 496$
5	$2^5 3 \cdot 7 = 672$
6	$2^6 127 = 8128$, $2^6 13 \cdot 127 = 105664$
7	none
8	see Table 1b
9–11	none

TABLE 1b

All $n = 2^8 m \in \mathcal{H}$ *with* m *odd and squarefree*	$H(n)$
$2^8 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 23 \cdot 37 \cdot 43 \cdot 73 = 15007087898880$	989
$2^8 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 23 \cdot 37 \cdot 73 = 349002044160$	506
$2^8 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 37 \cdot 43 \cdot 73 = 652482082560$	516
$2^8 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 37 \cdot 43 \cdot 73 \cdot 257 = 167687895217920$	1028
$2^8 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 37 \cdot 43 \cdot 73 \cdot 1031 = 672709027119360$	1031
$2^8 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 37 \cdot 47 \cdot 73 = 713178090240$	517
$2^8 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 37 \cdot 47 \cdot 73 \cdot 1033 = 736712967217920$	1033
$2^8 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 37 \cdot 73 = 15174001920$	264
$2^8 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 37 \cdot 73 \cdot 131 = 1987794251520$	524
$2^8 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 37 \cdot 73 \cdot 131 \cdot 523 = 1039616393544960$	1046
$2^8 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 37 \cdot 73 \cdot 263 = 3990762504960$	526
$2^8 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 37 \cdot 73 \cdot 263 \cdot 1051 = 4194291392712960$	1051
$2^8 3 \cdot 5 \cdot 7 \cdot 17 \cdot 19 \cdot 31 \cdot 37 \cdot 61 \cdot 73 = 44345330883840$	1037
$2^8 3 \cdot 5 \cdot 7 \cdot 17 \cdot 19 \cdot 31 \cdot 37 \cdot 73 = 726972637440$	527
$2^8 3 \cdot 5 \cdot 7 \cdot 17 \cdot 19 \cdot 37 \cdot 67 \cdot 73 = 1571198926080$	536
$2^8 3 \cdot 5 \cdot 7 \cdot 17 \cdot 19 \cdot 37 \cdot 73 = 23450730240$	272
$2^8 3 \cdot 5 \cdot 7 \cdot 17 \cdot 19 \cdot 37 \cdot 73 \cdot 271 = 6355147895040$	542
$2^8 3 \cdot 5 \cdot 7 \cdot 17 \cdot 19 \cdot 37 \cdot 73 \cdot 271 \cdot 541 = 3438135011216640$	1082
$2^8 3 \cdot 5 \cdot 7 \cdot 19 \cdot 23 \cdot 37 \cdot 73 = 31727458560$	276
$2^8 3 \cdot 5 \cdot 7 \cdot 19 \cdot 23 \cdot 37 \cdot 73 \cdot 137 = 4346661822720$	548
$2^8 3 \cdot 5 \cdot 7 \cdot 19 \cdot 23 \cdot 37 \cdot 73 \cdot 137 \cdot 547 = 2377624017027840$	1094
$2^8 3 \cdot 5 \cdot 7 \cdot 19 \cdot 23 \cdot 37 \cdot 73 \cdot 137 \cdot 547 \cdot 1093 = 2598743050611429120$	2186
$2^8 3 \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 37 \cdot 61 \cdot 73 = 2608548875520$	549
$2^8 3 \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 37 \cdot 61 \cdot 73 \cdot 1097 = 2861578116445440$	1097
$2^8 3 \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 37 \cdot 73 = 42763096320$	279
$2^8 3 \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 37 \cdot 73 \cdot 557 = 23819044650240$	557
$2^8 3 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 47 \cdot 73 = 64834371840$	282
$2^8 3 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 47 \cdot 73 \cdot 281 = 18218458487040$	562
$2^8 3 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 47 \cdot 73 \cdot 281 \cdot 1123 = 20459328880945920$	1123
$2^8 3 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 47 \cdot 73 \cdot 563 = 36501751345920$	563
$2^8 3 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 71 \cdot 73 = 97941285120$	284
$2^8 3 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 71 \cdot 73 \cdot 283 = 27717383688960$	566
$2^8 3 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73 = 1379454720$	144
$2^8 5 \cdot 7 \cdot 11 \cdot 19 \cdot 31 \cdot 37 \cdot 61 \cdot 73 = 9564679210240$	671
$2^8 5 \cdot 7 \cdot 11 \cdot 19 \cdot 31 \cdot 37 \cdot 73 = 156798019840$	341
$2^8 5 \cdot 7 \cdot 11 \cdot 19 \cdot 37 \cdot 43 \cdot 73 = 217494027520$	344
$2^8 5 \cdot 7 \cdot 11 \cdot 19 \cdot 37 \cdot 73 = 5058000640$	176
$2^8 5 \cdot 7 \cdot 19 \cdot 23 \cdot 37 \cdot 73 = 10575819520$	184
$2^8 5 \cdot 7 \cdot 19 \cdot 23 \cdot 37 \cdot 73 \cdot 367 = 3881325763840$	367
$2^8 5 \cdot 7 \cdot 19 \cdot 23 \cdot 37 \cdot 73 \cdot 367 \cdot 733 = 2845011784894720$	733
$2^8 5 \cdot 7 \cdot 19 \cdot 31 \cdot 37 \cdot 61 \cdot 73 = 869516291840$	366
$2^8 5 \cdot 7 \cdot 19 \cdot 31 \cdot 37 \cdot 73 = 14254365440$	186
$2^8 5 \cdot 7 \cdot 19 \cdot 37 \cdot 47 \cdot 73 = 21611457280$	188
$2^8 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73 = 459818240$	96
$2^8 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73 \cdot 191 = 87825283840$	191

with the following exceptions (in which p *denotes a prime):* $n = p$, $n = 2p$, $n = 6p$ $(p \neq 3)$, $n = 30p$ $(7 \leq p \leq 23)$, *and* $n = 1$, 15, 21, 70.

Theorem 3. *The only harmonic numbers* n *with* $H(n) \leq 13$ *are the thirteen apparent in Table* 3 (*see* §6).

2.

Throughout this note, roman letters denote positive integers and p and q denote primes. The canonical decomposition of n as a product of primes will always be written as

$$n = \prod_{i=1}^{t} p_i^{a_i},$$

where we assume that $p_1 < p_2 < \cdots < p_t$. To indicate that $p_i^{a_i} \mid n$, but $p_i^{a_i+1} \nmid n$, we write $p_i^{a_i} \parallel n$.

We set $S(n) = \sigma(n)/n$. Then

$$\lim_{b \to \infty} S(p^b) = \lim_{b \to \infty} \frac{p^{b+1} - 1}{p^b(p-1)} = \frac{p}{p-1};$$

for convenience, this limit will be denoted by $S(\overline{p})$. If $p < q$, $a < b$ and $c \geq 1$, then it is easily verified that

$$1 < S(q^a) < S(q^b) < S(\overline{q}) = \frac{q}{q-1} \leq \frac{p+1}{p} = S(p) \leq S(p^c). \tag{2}$$

Recall that σ and τ are multiplicative functions, so that H and S are also. We specify further that S is multiplicative in an extended sense whereby we can write for example $S(p^a\overline{q}) = S(p^a)S(\overline{q})$, for distinct primes p and q. (Consider the left-hand side of this equation to be defined by the right-hand side.)

We will denote by $\mathcal{H}$ the set of all harmonic numbers.

The following lemmas are required.

Lemma 1. *Besides* 1, *the only squarefree harmonic number is* 6.

Lemma 2. *There are no harmonic numbers of the form* p^a.

Lemma 3. *The only harmonic numbers of the form* $p^a q^b$ *are perfect numbers.*

Lemma 4. *If* $n = 2^{a-1}(2^a - 1)$ *is perfect* (*so that* $2^a - 1$ *and* a *are primes*), *then* $H(n) = a$.

Lemmas 1 and 2 are due to Ore [7]. Lemma 3 was first proved by Pomerance [8] and was rediscovered by Callan [1]. Lemma 4 is easily proved, assuming some elementary knowledge of perfect numbers. Notice that Lemma 1 can be expressed equivalently as: If $n \in \mathcal{H}$ and $n > 6$, then $\tau(n) \geq 2^{\omega(n)-1}3$.

Lemma 5. *Let* a *be a positive integer such that, for some prime* $p \equiv 2 \pmod 3$, *we have* $3p \mid \sigma(2^a)$ *and* $(3p, a+1) = 1$. *If* m *is an odd integer such that* $2^a m$ *is harmonic, then* m *is not squarefree.*

Proof. Note that $p \neq 2$. Let m be an odd integer such that $n = 2^a m$ is harmonic, and suppose further that m is squarefree. Then $\tau(n) = 2^{\omega(m)}(a+1)$. Since $H(n)\sigma(n) = n\tau(n)$ and $3p \mid \sigma(n)$ and $(3p, a+1) = 1$, we have $3 \parallel n$ and $p \parallel n$. The former implies that 3 exactly divides the right-hand side of the equation $H(n)\sigma(n) = n\tau(n)$, and the latter implies that 3^2 divides the left-hand side, since $3 \mid (p+1) \mid \sigma(n)$ and $3 \mid \sigma(2^a)$. This contradiction shows that m cannot be squarefree. $\square$

Lemma 6. *Suppose n is harmonic. If p is an odd prime such that $p \parallel n$ and $p \mid H(n)$, then n/p is harmonic. If $2 \parallel n$ and $4 \mid H(n)$, then $n/2$ is harmonic.*

The proof is straightforward.

Lemma 7. *If n is harmonic and $H(n) = p$, a prime, then either $p \mid n$ or n is perfect.*

Proof. Since $H(n)\sigma(n) = n\tau(n)$, we have $H(n) = p \mid n\tau(n)$. Suppose $p \mid \tau(n)$. Then $q^{p-1} \mid n$ for some prime q and so $\tau(n) \geq 2^{\omega(n)-1}p$. Write $n = \prod_{i=1}^{\omega(n)} p_i^{a_i}$, as above. Then $p_i \geq i+1$ for all i and

$$\tau(n) = H(n)S(n) = p\prod_{i=1}^{\omega(n)} S\left(p_i^{a_i}\right) < p\prod_{i=1}^{\omega(n)} \frac{p_i}{p_i - 1} \leq p\prod_{i=1}^{\omega(n)} \frac{i+1}{i} = p(\omega(n)+1).$$

It follows that $\omega(n)+1 > 2^{\omega(n)-1}$, and this is a contradiction when $\omega(n) \geq 3$. Thus, $p \mid n$ when $\omega(n) \geq 3$. Otherwise, by Lemmas 2, 3 and 4, n is perfect. $\square$

Lemma 8. *Suppose $\omega(n) = 3$ or 4. Then $\sigma(n) \neq 2n$. If $\sigma(n) = 3n$ then $n \in \{120, 672, 523776\}$.*

The proof is omitted. It uses results of Hagis [5] and Steuerwald [9].

Lemma 9. *Let n be an odd harmonic number. If $p^a \parallel n$, then $p^a \equiv 1 \pmod 4$.*

This result is Theorem 2 in Garcia [3]. It was derived independently, and stated in the form above, by Mills [6].

3.

Sketch of the proof of Theorem 1. For each value of a in turn, we put $n = 2^a m$, where m is odd and squarefree, and assume $n \in \mathcal{H}$. The case $a = 1$ is clear from Lemma 1, and Lemma 5 accounts for the cases $a = 3$, 7 and 9, because $\sigma(2^3) = 3\cdot 5$, $\sigma(2^7) = 3\cdot 5\cdot 17$ and $\sigma(2^9) = 3\cdot 11\cdot 31$.

The following notation is convenient. For any prime q and distinct primes q_1, $\dots$, q_s, write

$$q' = \frac{\sigma(q)}{\tau(q)} = \frac{q+1}{2}, \qquad Q_{j,s} = \prod_{i=j}^{s} \frac{q_i}{q_i'},\ 1 \leq j \leq s.$$

Notice that $Q_{j,s} = H(q_j \dots q_s)$.

Suppose $a = 2$. Since $H(n)\sigma(n) = n\tau(n)$, we have $\sigma(2^2) = 7 \parallel n$. Set $n = 2^2 7k$, where $(k, 14) = 1$, and note that $H(2^2 7) = 3$. Either $k = 1$, giving $n = 2^2 7 = 28$, or k is squarefree and $3H(k)$ is an integer. In this case, set $k = q_1 \dots q_s$, where q_1, $\dots$, q_s are distinct primes, not 2 or 7. We have

$$H(n) = 3H(k) = \frac{3q_1 \dots q_s}{q_1' \dots q_s'} = 3Q_{1,s}.$$

The argument from here, and similarly in the remainder of this proof, rests on determining whether and how, in this rational number, the denominator can fully factor into the numerator to produce an integer. We take $q_1' = 3$ so that $q_1 = 5$. If

$s = 1$, then we have found $n = 2^2 5 \cdot 7 = 140$. Otherwise, $H(n) = 5Q_{2,s}$, in which now we may suppose that $q_2 < \cdots < q_s$. We cannot have $q_2' = 5$, since then $q_2 = 9$ is not prime, and $q_2' < q_i$ for $i \geq 2$, so we have found all such $n = 2^2 m$.

The proofs for the cases $a = 4$, 5, 6, 10 and 11 are similarly short, and will not be given here, but that for the case $a = 8$ is very long. It begins as follows.

Suppose $n = 2^8 m$. Since $\sigma(2^8) = 7 \cdot 73$ and $H(n)\sigma(n) = n\tau(n)$ with $\tau(n) = 2^{\omega(n)-1}3^2$, we have $7 \parallel n$ and $73 \parallel n$. Put $n = 2^8 7 \cdot 73k$, where $k = q_1 \dots q_s$, a product of distinct primes, not 2, 7 or 73. We have

$$H(2^8 7 \cdot 73) = \frac{2^6 3^2}{37},$$

and so we require $H(n) = (2^6 3^2/37)H(k) = (2^6 3^2/37)Q_{1,s}$ to be an integer. Take $q_1 = 37$. Then $q_1' = 19$ and $H(n) = (2^6 3^2/19)Q_{2,s}$. Take $q_2 = 19$, so $q_2' = 10$ and $H(n) = (2^5 3^2/5)Q_{3,s}$. Take $q_3 = 5$, so $q_3' = 3$ and $H(n) = 2^5 3 Q_{4,s}$.

If $s = 3$, then we have in fact found the solution $n = 2^8 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73$. All others arise from this "seed" and depend on finding a value for q_i' which divides the numerical part of the current numerator and for which q_i is a prime different from those already encountered. It is easy to verify that harmonic numbers cannot otherwise arise.

There are the following possibilities for q_4': 2, 2^4, $2 \cdot 3$, $2^2 3$, $2^3 3$ and $2^5 3$. (These are the acceptable divisors of $2^5 3$.) If $q_4' = 2$, then $q_4 = 3$ and $H(n) = 2^4 3^2 Q_{5,s}$. If $s = 4$, then we have found the solution $n = 2^8 3 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73$. Otherwise, there are the following possibilities for q_5': 2^4, $2 \cdot 3$, $2^2 3$, $2^3 3$, $2^2 3^2$ and 3^2. (These are the acceptable divisors of $2^4 3^2$.) If $q_5' = 2^4$, then $q_5 = 31$ and $H(n) = 3^2 31 Q_{6,s}$. If $s = 5$, then we have found the solution $n = 2^8 3 \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 37 \cdot 73$. Otherwise, we must consider possible values of q_6', and so on. Then we must consider the other possible values of q_5', and then the other possible values of q_4'.

The proof continues until all possibilities have been considered. □

4.

Sketch of the proof of Theorem 2. Suppose $n > 1$ and, as above, write $n = \prod_{i=1}^t p_i^{a_i}$. Since $p_i \geq i + 1$ for $1 \leq i \leq t$, we have

$$\frac{\sigma(n)}{n} < \prod_{i=1}^t \frac{p_i}{p_i - 1} \leq \prod_{i=1}^t \frac{i+1}{i} = t + 1.$$

Then, since $H(n) = \tau(n)/(\sigma(n)/n)$, the theorem is proved for those n for which $\tau(n) \geq 2^{t+1}$. In fact, this is always the case, except if n is squarefree or of the form $p^2 m$, where m is squarefree and $p \nmid m$. These possibilities must be investigated more closely, leading to the exceptions noted in the statement of the theorem. □

An easy consequence of Theorem 2, using Lemmas 1, 2 and 3, is that the inequality (1) holds for any harmonic number n, $n > 6$.

For our proof of Theorem 3, we require a tabulated improvement of (1), for harmonic numbers n with small values of $\omega(n)$. Let P_i denote the ith prime, so that $P_1 = 2$, $P_2 = 3$, $\dots$. Then $p_i \geq P_i$ for each i, and our improvement is based on exact calculations with $P_i/(P_i - 1)$, rather than $(i + 1)/i$ as in the above proof.

For example, if $2 \parallel n$ and $\omega(n) = 3$, then using (2), $S(n) = S(2p_2^{a_2}p_3^{a_3}) < S(2 \cdot \overline{3} \cdot \overline{5})$, so that

$$H(n) = \frac{\tau(n)}{\sigma(n)/n} = \frac{\tau(n)}{S(n)} > \frac{2^2 3}{(3/2)(3/2)(5/4)} = \frac{64}{15}.$$

In this way, we have constructed the column of lower bounds for $H(n)$ headed $2 \parallel n$ in Table 2. For each value of $\omega(n)$, $H(n)$ is not less than the corresponding entry in this column. The other columns treat the special cases in which $2^a \parallel n$ for $2 \le a \le 11$, $2^{12} \mid n$ and $2 \nmid n$, respectively.

TABLE 2

	Lower bounds for $H(n)$ if $n \in \mathcal{H}$					
$\omega(n)$	$2 \parallel n$	$2^2 \parallel n$	$2^3 \parallel n$	$2^4 \parallel n$	$2^5 \parallel n$	$2^6 \parallel n$
3	5	6[a]	7	10	11[b]	14[c]
4	8	10	12	16	17	23
5	14	18	22	28	31	39
6	25	32	40	50	57	70
7	47	60	74	92	106	129
≥ 8	88	113	141	174	201	243

[a] $n \ne 140$, [b] $n \ne 672$, [c] $n \ne 105664$

	Lower bounds for $H(n)$ if $n \in \mathcal{H}$						
$\omega(n)$	$2^7 \parallel n$	$2^8 \parallel n$	$2^9 \parallel n$	$2^{10} \parallel n$	$2^{11} \parallel n$	$2^{12} \mid n$	$2 \nmid n$
3	16	23	27	32	23	14	6
4	25	31	36	42	36	24	10
5	42	49	57	67	61	44	19
6	76	89	97	115	111	80	35
7	140	164[d]	179	208	209	151	66
≥ 8	264	264	336	384	395	285	126

[d] $n \ne 1379454720$

Suppose $2^2 \parallel n$, $\omega(n) \ge 3$ and $n \ne 140$. By Theorem 1, $\tau(n) \ge 2^{\omega(n)-2}3^2$. Also, $S(n) < S(2^2)\prod_{i=2}^{\omega(n)} S(\overline{P}_i)$. This is used in the calculations for the third column of Table 2, and in this fashion Table 2 may be completed.

Notice from the table that $\omega(n) \le 4$ if $H(n) \le 13$. Not having to consider $\omega(n) \ge 5$ was the main reason for seeking only those $n \in \mathcal{H}$ with $H(n) \le 13$ in Theorem 3. The number of columns in Table 2 was determined by continuing until it could be asserted that if n is harmonic with $H(n) \le 13$, and n is even, then $2^6 \nmid n$ (except if $n = 105664$).

5.

The proof of Theorem 3 will be described in this section. There will be many applications of Lemma 1 and (2), often without special mention. Lemma 1 implies that, for $n \in \mathcal{H}$, $\tau(n) \ne 8$ if $\omega(n) = 3$, and $\tau(n) \ne 16$ if $\omega(n) = 4$. For reference throughout the full proof, [2] includes a convenient table of possible values of $\tau(n)$ for $\omega(n) = 3$ and 4, together with the corresponding possible exponents on the prime factors of n.

The proof makes considerable use of the following technical lemma, whose proof we omit.

Lemma 10. *Suppose n is a harmonic number, satisfying $n > 2 \cdot 10^9$ and $H(n) \leq 13$. Then n has a prime factor exceeding 20.*

Sketch of the proof of Theorem 3. As a function of $\omega(n)$, the right-hand side of (1) is increasing, and equals 4 when $\omega(n) = 3$. Therefore, Theorem 2 implies that $H(n) \geq 5$ when $\omega(n) \geq 3$. Then if $H(n) \leq 4$ we must have $n = 1$, since $H(1) = 1$, or $\omega(n) = 2$, by Lemma 2. Then, by Lemmas 3 and 4, $n = 6$ or 28. In particular, there is no solution of the equation $H(n) = 4$.

Suppose $H(n) = 5$, so that $5\sigma(n) = n\tau(n)$. Two solutions, $n = 140$ and $n = 496$, are evident from Table 3. For any other solution, from Table 2 we have $\omega(n) = 3$ and $2 \parallel n$, and from Lemma 7 we have $5 \mid n$. By Lemma 10, the remaining prime factor exceeds 20. Then $\tau(n) = 5S(n) < 5S(2 \cdot \overline{5} \cdot \overline{23}) < 9.9$, a contradiction of Lemma 1.

The proof continues in this manner, considering in turn each possible value of $H(n)$. Each case requires a special argument and, generally speaking, each is more complicated than the one before. Here, we shall give further only the penultimate case.

Suppose $H(n) = 12$, so that $12\sigma(n) = n\tau(n)$. Using Lemma 6, we deduce that we cannot have $2 \parallel n$ or $3 \parallel n$, and, from Lemma 8, $\tau(n) \neq 24$ or 36. Of course, $\tau(n) \neq 12$. We note that there are no solutions in Table 3.

Suppose n is odd. Then we cannot have $\omega(n) = 4$ since this implies $32 \leq \tau(n) = 12S(n) < 12S(\overline{3} \cdot \overline{5} \cdot \overline{7} \cdot \overline{23}) < 27.5$. So, if n is odd, then $\omega(n) = 3$, and $\tau(n) < 12S(\overline{3} \cdot \overline{5} \cdot \overline{23}) < 23.6$. Notice that $4 \mid \tau(n)$, since otherwise $2 \mid n$, so then $3 \nmid \tau(n)$. Thus $3^2 \mid n$, but we cannot have $3^2 \parallel n$, else $3 \mid \tau(n)$, or $3^3 \parallel n$, by Lemma 9. The only possibility is then $n = 3^4pq$, for distinct odd primes p, q exceeding 3; but then we must also have $\sigma(3^4) = 11^2 \mid n$, a contradiction.

Suppose n is even and $\omega(n) = 4$. From Table 2, $2^4 \nmid n$, so $\tau(n) < 12S(2^3\overline{3} \cdot \overline{5} \cdot \overline{23}) < 44.2$. The only possible exponents for the four prime factors of n are arrangements of 3, 1, 1, 1 or 4, 1, 1, 1. Since $2^2 \mid n$, these arrangements are impossible, by Theorem 1. Hence $\omega(n) = 3$.

From Table 2, $2^6 \nmid n$, so that $\tau(n) < 12S(2^5\overline{3} \cdot \overline{23}) < 37.1$. Suppose first that $3 \mid n$. Then, since $2^2 3^2 \mid n$, n must equal $2^3 3^3 p$, $2^4 3^2 p$, $2^2 3^4 p$, $2^2 3^2 p^2$ or $2^2 3^2 p$, all of which are easily eliminated, whatever the prime p.

Then we may now suppose that $2^2 \mid n$, $3 \nmid n$ and $\omega(n) = 3$. Then $3 \mid \tau(n)$ and $\tau(n) < 12S(2^5\overline{5} \cdot \overline{23}) < 30.9$. It follows, using Theorem 1, that we cannot have $2^5 \parallel n$ or $2^3 \parallel n$. If $2^4 \parallel n$, then $\sigma(2^4) = 31 \mid n$, and either $n = 2^4 31 p^2$ or $n = 2^4 31^2 p$; it is easy to see that neither can be harmonic, for any p. If $2^2 \parallel n$, then $\sigma(2^2) = 7 \mid n$, but $7^5 \nmid n$. We cannot have $7^4 \parallel n$, since then $\sigma(7^4) = 2801 \parallel n$, but $2^2 7^4 2801 \notin \mathcal{H}$; we cannot have $7^3 \parallel n$, since then $12 \mid \tau(n)$; we cannot have $7^2 \parallel n$, since then $\frac{1}{3}\sigma(7^2) = 19 \mid n$, contradicting Lemma 10; we cannot have $7 \parallel n$, since then $n = 2^2 7 p^a$ for $a = 2$ or 4 and $(14, p) = 1$, but substituting this into $12\sigma(n) = n\tau(n)$ leads to $4 \mid (a+1)$, a contradiction. □

6.

In Table 3 we list all 130 harmonic numbers not exceeding $2 \cdot 10^9$. The next smallest harmonic number is $2008725600 = 2^5 3 \cdot 5^2 7^2 19 \cdot 29 \cdot 31$. A very simple search procedure, written in UBASIC, was allowed to run a very long time to produce this table.

TABLE 3

All $n \in \mathcal{H}$, $1 \le n \le 2 \cdot 10^9$			
n	$H(n)$	n	$H(n)$
1	1	$56511000 = 2^3 3^3 5^3 7 \cdot 13 \cdot 23$	115
$6 = 2 \cdot 3$	2	$69266400 = 2^5 3 \cdot 5^2 7^2 19 \cdot 31$	105
$*28 = 2^2 7$	3	$71253000 = 2^3 3^3 5^3 7 \cdot 13 \cdot 29$	116
$140 = 2^2 5 \cdot 7$	5	$75038600 = 2^3 5^2 7^2 13 \cdot 19 \cdot 31$	91
$270 = 2 \cdot 3^3 5$	6	$80832960 = 2^6 3^2 5 \cdot 13 \cdot 17 \cdot 127$	85
$*496 = 2^4 31$	5	$81695250 = 2 \cdot 3^3 5^3 7^2 13 \cdot 19$	105
$672 = 2^5 3 \cdot 7$	8	$90409410 = 2 \cdot 3^2 5 \cdot 7^2 13 \cdot 19 \cdot 83$	83
$1638 = 2 \cdot 3^2 7 \cdot 13$	9	$108421632 = 2^9 3^3 11 \cdot 23 \cdot 31$	92
$2970 = 2 \cdot 3^3 5 \cdot 11$	11	$110583200 = 2^5 5^2 7^3 13 \cdot 31$	91
$6200 = 2^3 5^2 31$	10	$115048440 = 2^3 3^2 5 \cdot 13^2 31 \cdot 61$	78
$*8128 = 2^6 127$	7	$115462620 = 2^2 3^2 5 \cdot 7^2 13 \cdot 19 \cdot 53$	106
$8190 = 2 \cdot 3^2 5 \cdot 7 \cdot 13$	15	$137891520 = 2^6 3^2 5 \cdot 13 \cdot 29 \cdot 127$	87
$18600 = 2^3 3 \cdot 5^2 31$	15	$*142990848 = 2^9 3^2 7 \cdot 11 \cdot 13 \cdot 31$	120
$18620 = 2^2 5 \cdot 7^2 19$	14	$144963000 = 2^3 3^3 5^3 7 \cdot 13 \cdot 59$	118
$27846 = 2 \cdot 3^2 7 \cdot 13 \cdot 17$	17	$163390500 = 2^2 3^3 5^3 7^2 13 \cdot 19$	135
$30240 = 2^5 3^3 5 \cdot 7$	24	$164989440 = 2^9 3^3 5 \cdot 7 \cdot 11 \cdot 31$	140
$32760 = 2^3 3^2 5 \cdot 7 \cdot 13$	24	$191711520 = 2^5 3^2 5 \cdot 7^2 11 \cdot 13 \cdot 19$	176
$55860 = 2^2 3 \cdot 5 \cdot 7^2 19$	21	$221557248 = 2^9 3^3 11 \cdot 31 \cdot 47$	94
$105664 = 2^6 13 \cdot 127$	13	$233103780 = 2^2 3^2 5 \cdot 7^2 13 \cdot 19 \cdot 107$	107
$117800 = 2^3 5^2 19 \cdot 31$	19	$255428096 = 2^9 7 \cdot 11^2 19 \cdot 31$	88
$167400 = 2^3 3^3 5^2 31$	27	$287425800 = 2^3 3^3 5^2 17 \cdot 31 \cdot 101$	101
$173600 = 2^5 5^2 7 \cdot 31$	25	$300154400 = 2^5 5^2 7^2 13 \cdot 19 \cdot 31$	130
$237510 = 2 \cdot 3^2 5 \cdot 7 \cdot 13 \cdot 29$	29	$*301953024 = 2^{12} 3^2 8191$	27
$242060 = 2^2 5 \cdot 7^2 13 \cdot 19$	26	$318177800 = 2^3 5^2 19 \cdot 31 \cdot 37 \cdot 73$	73
$332640 = 2^5 3^3 5 \cdot 7 \cdot 11$	44	$318729600 = 2^7 3^3 5^2 7 \cdot 17 \cdot 31$	168
$360360 = 2^3 3^2 5 \cdot 7 \cdot 11 \cdot 13$	44	$326781000 = 2^3 3^3 5^3 7^2 13 \cdot 19$	168
$539400 = 2^3 3 \cdot 5^2 29 \cdot 31$	29	$400851360 = 2^5 3^2 5 \cdot 7^2 13 \cdot 19 \cdot 23$	184
$695520 = 2^5 3^3 5 \cdot 7 \cdot 23$	46	$407386980 = 2^2 3^2 5 \cdot 7^2 11 \cdot 13 \cdot 17 \cdot 19$	187
$726180 = 2^2 3 \cdot 5 \cdot 7^2 13 \cdot 19$	39	$423184320 = 2^6 3^2 5 \cdot 13 \cdot 89 \cdot 127$	89
$753480 = 2^3 3^2 5 \cdot 7 \cdot 13 \cdot 23$	46	$428972544 = 2^9 3^3 7 \cdot 11 \cdot 13 \cdot 31$	156
$*950976 = 2^6 3^2 13 \cdot 127$	27	$447828480 = 2^9 3^3 5 \cdot 11 \cdot 19 \cdot 31$	152
$1089270 = 2 \cdot 3^2 5 \cdot 7^2 13 \cdot 19$	42	$*459818240 = 2^8 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73$	96
$1421280 = 2^5 3^3 5 \cdot 7 \cdot 47$	47	$481572000 = 2^5 3^3 5^3 7^3 13$	168
$1539720 = 2^3 3^2 5 \cdot 7 \cdot 13 \cdot 47$	47	$499974930 = 2 \cdot 3^5 5 \cdot 7^2 13 \cdot 17 \cdot 19$	153
$*2178540 = 2^2 3^2 5 \cdot 7^2 13 \cdot 19$	54	$500860800 = 2^7 3^3 5^2 11 \cdot 17 \cdot 31$	176
$2229500 = 2^2 5^3 7^3 13$	35	$513513000 = 2^3 3^3 5^3 7 \cdot 11 \cdot 13 \cdot 19$	209
$2290260 = 2^2 3 \cdot 5 \cdot 7^2 19 \cdot 41$	41	$526480500 = 2^2 3 \cdot 5^3 7^2 13 \cdot 19 \cdot 29$	145
$2457000 = 2^3 3^3 5^3 7 \cdot 13$	60	$540277920 = 2^5 3^2 5 \cdot 7^2 13 \cdot 19 \cdot 31$	186
$2845800 = 2^3 3^3 5^2 17 \cdot 31$	51	$559903400 = 2^3 5^2 7^2 19 \cdot 31 \cdot 97$	97
$4358600 = 2^3 5^2 19 \cdot 31 \cdot 37$	37	$623397600 = 2^5 3^3 5^2 7^2 19 \cdot 31$	189
$4713984 = 2^9 3^3 11 \cdot 31$	48	$644271264 = 2^5 3^2 7 \cdot 13^2 31 \cdot 61$	117
$4754880 = 2^6 3^2 5 \cdot 13 \cdot 127$	45	$*675347400 = 2^3 3^2 5^2 7^2 13 \cdot 19 \cdot 31$	189
$5772200 = 2^3 5^2 7^2 19 \cdot 31$	49	$*714954240 = 2^9 3^2 5 \cdot 7 \cdot 11 \cdot 13 \cdot 31$	200
$6051500 = 2^2 5^3 7^2 13 \cdot 19$	50	$758951424 = 2^9 3^3 7 \cdot 11 \cdot 23 \cdot 31$	161
$8506400 = 2^5 5^2 7^3 31$	49	$766284288 = 2^9 3 \cdot 7 \cdot 11^2 19 \cdot 31$	132
$8872200 = 2^3 3^3 5^2 31 \cdot 53$	53	$819131040 = 2^5 3^2 5 \cdot 7^2 13 \cdot 19 \cdot 47$	188
$11981970 = 2 \cdot 3^2 5 \cdot 7^2 11 \cdot 13 \cdot 19$	77	$825120800 = 2^5 5^2 7^3 31 \cdot 97$	97
$14303520 = 2^5 3^3 5 \cdot 7 \cdot 11 \cdot 43$	86	$886402440 = 2^3 3^4 5 \cdot 7 \cdot 11^2 17 \cdot 19$	204
$15495480 = 2^3 3^2 5 \cdot 7 \cdot 11 \cdot 13 \cdot 43$	86	$900463200 = 2^5 3 \cdot 5^2 7^2 13 \cdot 19 \cdot 31$	195
$16166592 = 2^6 3^2 13 \cdot 17 \cdot 127$	51	$*995248800 = 2^5 3^2 5^2 7^3 13 \cdot 31$	189
$17428320 = 2^5 3^2 5 \cdot 7^2 13 \cdot 19$	96	$1047254400 = 2^7 3^3 5^2 17 \cdot 23 \cdot 31$	184
$18154500 = 2^2 3 \cdot 5^3 7^2 13 \cdot 19$	75	$1162161000 = 2^3 3^3 5^3 7 \cdot 11 \cdot 13 \cdot 43$	215
$23088800 = 2^5 5^2 7^2 19 \cdot 31$	70	$1199250360 = 2^3 3^4 5 \cdot 7 \cdot 11^2 19 \cdot 23$	207
$23569920 = 2^9 3^3 5 \cdot 11 \cdot 31$	80	$1265532840 = 2^3 3^2 5 \cdot 11 \cdot 13^2 31 \cdot 61$	143
$23963940 = 2^2 3^2 5 \cdot 7^2 11 \cdot 13 \cdot 19$	99	$1307124000 = 2^5 3^3 5^3 7^2 13 \cdot 19$	240
$27027000 = 2^3 3^3 5^3 7 \cdot 11 \cdot 13$	110	$1352913408 = 2^9 3^3 7 \cdot 11 \cdot 31 \cdot 41$	164
$29410290 = 2 \cdot 3^5 5 \cdot 7^2 13 \cdot 19$	81	$*1379454720 = 2^8 3 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73$	144
$32997888 = 2^9 3^3 7 \cdot 11 \cdot 31$	84	$1381161600 = 2^7 3^2 5^2 7 \cdot 13 \cdot 17 \cdot 31$	240
$*33550336 = 2^{12} 8191$	13	$1509765120 = 2^{12} 3^2 5 \cdot 8191$	45
$37035180 = 2^2 3^2 5 \cdot 7^2 13 \cdot 17 \cdot 19$	102	$1558745370 = 2 \cdot 3^5 5 \cdot 7^2 13 \cdot 19 \cdot 53$	159
$44660070 = 2 \cdot 3^2 5 \cdot 7^2 13 \cdot 19 \cdot 41$	82	$1630964808 = 2^3 3^4 11^3 31 \cdot 61$	99
$45532800 = 2^7 3^3 5^2 17 \cdot 31$	96	$1632825792 = 2^6 3^2 13 \cdot 17 \cdot 101 \cdot 127$	101
$46683000 = 2^3 3^3 5^3 7 \cdot 13 \cdot 19$	114	$1727271000 = 2^3 3^3 5^3 7 \cdot 13 \cdot 19 \cdot 37$	222
$50401728 = 2^6 3^2 13 \cdot 53 \cdot 127$	53	$1862023680 = 2^9 3^3 5 \cdot 11 \cdot 31 \cdot 79$	158
$52141320 = 2^3 3^4 5 \cdot 7 \cdot 11^2 19$	108	$1867650048 = 2^{10} 3^4 11 \cdot 23 \cdot 89$	128

Except for those marked with an asterisk, all numbers in Table 3 are also arithmetic: a natural number n is arithmetic if the arithmetic mean $A(n)$ of its positive divisors is an integer. In [7], Ore describes his interest in harmonic numbers that are neither arithmetic nor perfect; he originally thought perhaps there were none, but found the first example himself.

It is easy to see that $A(n)H(n) = n$ for any n, so those numbers marked with an asterisk in Table 3 are, equivalently, those $n \in \mathcal{H}$, $n \leq 2 \cdot 10^9$, for which $H(n) \nmid n$. There are six other examples in Table 1b.

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